
Numerical Analysis and Computational Mathematics

Fall Semester 2024 – CSE Section

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Session 11 – November 27, 2024

Solutions – Linear systems & Eigenproblems

Solution I (MATLAB)

- a) We assign the matrix A , visualize the truss bridge in Figure 1, and plot the pattern of A in Figure 2(left) by means of the following commands.

```
KK = 1e3;  
nnodes = 29; % number of nodes of the bridge (odd)  
[ A ] = bridge_stiffness_matrix( nnodes, KK );  
plot_bridge( nnodes );  
spy( A );
```

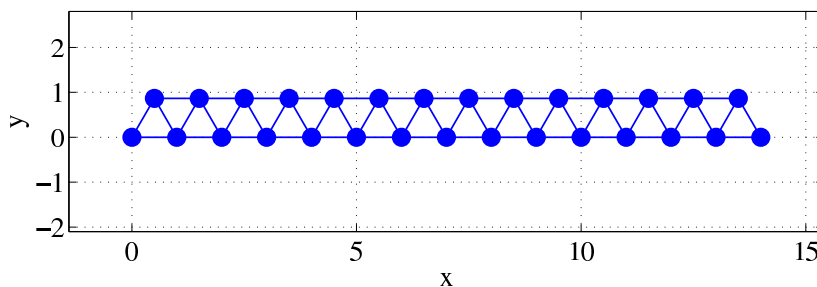


Figure 1: Truss bridge with $N_{nodes} = 29$.

We observe that the matrix A is sparse and banded. Then, we verify that the sparse matrix A is symmetric by plotting the pattern of the matrix $A - A'$ in Figure 2(right); we observe that each element of the matrix $A - A'$ is identically zero and thus A is symmetric.

```
spy( A' - A );
```

We verify that the matrix A is singular. Instead of computing the determinant of the matrix A , we estimate its conditioning number by means of the MATLAB command `condest` which returns the result `Inf`.

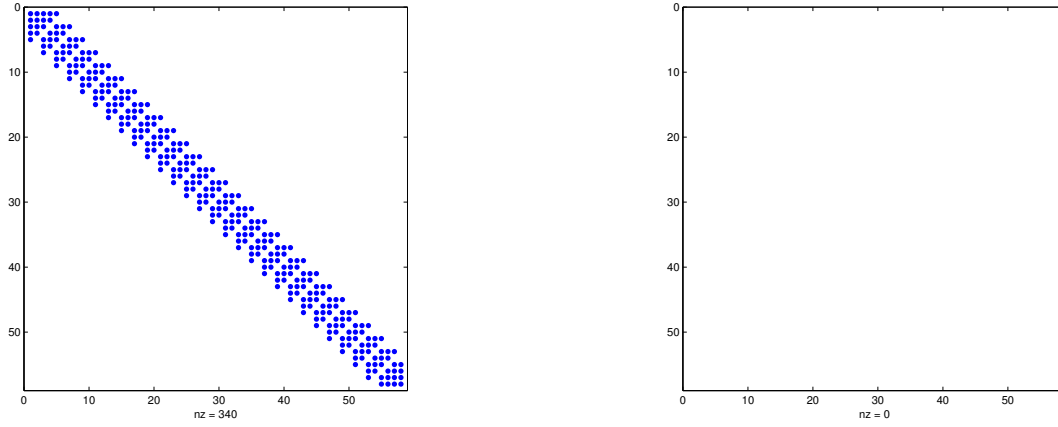


Figure 2: Patterns of the sparse matrices A (left) and $A - A'$ (right); the nonzero elements are indicated by dots.

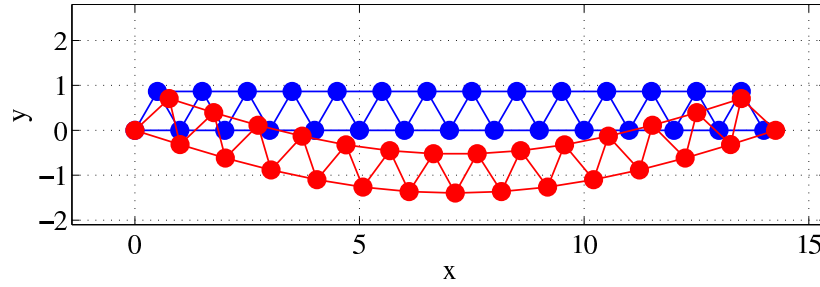


Figure 3: Truss bridge; deformed configuration (red) for the load $\mathbf{f}_i^{ext}$, $i = 1, 3, \dots, N_{nodes}$.

```
K_A = condest( A )
%   K_A =
%       Inf
```

- b) We impose the constraints to obtain the matrix $\tilde{A}$ from A and we verify that $\tilde{A}$ is nonsingular by means of the following commands. Ar corresponds to the matrix $\tilde{A}$.

```
n = size( A, 2 );
nr = n - 3;
ir = [ 3 : n - 1 ];
Ar = A( ir, ir );
K_Ar = condest( Ar )
%   K_Ar =
%       1.8838e+04
```

- c) We use the following MATLAB commands to obtain the result of Figure 3 representing the deformed configuration of the bridge; this is obtained by applying the displacement vector $\mathbf{d}$ to the nodes of the undeformed truss bridge of Figure 1.

```

node_force = 1 : 2 : nnodes;
b = zeros( n, 1 );
b( 2 * node_force, 1 ) = -1;
br = b( ir, 1 );
dr = Ar \ br;
d = zeros( n, 1 );
d( ir, 1 ) = dr;
[ fig_h ] = plot_bridge( nnodes, d );

```

br and dr correspond to the vectors $\tilde{\mathbf{b}}$ and $\tilde{\mathbf{d}}$, respectively.

- d) Since the LU factorization of the matrix $\tilde{A}$ is independent of the loads (the vector $\tilde{\mathbf{b}}$), we use the following MATLAB commands, with the factorization of the matrix $\tilde{A}$ performed only once.

```

[ Lr, Ur, Pr ] = lu( Ar );
for l = 1 : 10
    b = 2 * ( rand( n, 1 ) - 0.5 ); % random force in [-1,1]
    br = b( ir, 1 );
    [ yr ] = forward_substitutions( Lr, Pr * br );
    [ dr ] = backward_substitutions( Ur, yr );
    d = zeros( n, 1 );
    d( ir, 1 ) = dr;
    [ fig_h ] = plot_bridge( nnodes, d );
    pause( 1 );
end

```

- e) We use the following MATLAB commands to obtain that the GMRES and conjugate gradient methods converge to the solution, for the prescribed tolerance, in 54 and 56 iterations, respectively.

```

node_force = 1 : 2 : nnodes;
b = zeros( n, 1 );
b( 2 * node_force, 1 ) = -1;
br = b( ir, 1 );
dr_gmres = gmres( Ar, br, [], 5e-5, nr );
% gmres converged at iteration 54 to a solution with relative residual 1.8e-05.
dr_cg = pcg( Ar, br, 5e-5, 1000, [] );
% pcg converged at iteration 56 to a solution with relative residual 9.8e-06.

```

Solution II (MATLAB)

- a) We consider the following implementation of the MATLAB function power_method.m.

```

function [ lambda, x, k ] = power_method( A, x0, tol, kmax );
% POWER_METHOD power method for the computation of the largest eigenvalue
% (in modulus) of the matrix A (\lambda_1). We assume that A is square,
% |\lambda_1| > |\lambda_i| for i=2,...,n, and \lambda_1 non zero
% Stopping criterion based on the relative difference of successive
% iterates of the eigenvalue.

```

```

% [ lambda, x ] = power_method( A, x0, tol, kmax )
% Inputs: A      = matrix (n x n)
%          x0     = initial vector (n x 1)
%          tol    = tolerance for the stopping criterion
%          kmax   = maximum number of iterations
% Output: lambda = computed (largest) eigenvalue
%          x      = computed eigenvector corresponding to lambda
%          k      = number of iterations
%
x = x0;
y = x / norm( x );
lambda = y' * A * y;
err = tol + 1;
k = 0;
lambda_old = lambda;

while ( err >= tol && k <= kmax )

    x = A * y;
    y = x / norm( x );
    lambda = y' * A * y;

    err = abs( lambda - lambda_old ) / abs( lambda );

    lambda_old = lambda;
    k = k + 1;

end

return

```

- b) We consider the following MATLAB commands for which we obtain that $\lambda_1^{(k_c)} = 7.2589$ and $k_c = 23$.

```

n = 4;
A = [ 5 -2 -1 0; -2 5 -1 -1; -1 -1 4 -1; 0 -1 -1 5 ];
x0 = ones( n, 1 );
[ lambda1_c, x1, k_c ] = power_method( A, x0, 1e-6, 100 );
lambda1_c
%   lambda1_c =
%           7.2589
k_c
%   k_c =
%       23

```

We compute the relative error with respect to the exact value of λ_1 (computed by means of MATLAB). We obtain that $e^{(k_c)} = 7.9711 \cdot 10^{-7}$.

```

lambda1_ex = max( eig( A ) );
rel_err = abs( lambda1_c - lambda1_ex ) / abs( lambda1_ex )
%   rel_err =
%       7.9711e-07

```

- c) We consider the following MATLAB commands for which we obtain that $\lambda_n^{(k_c)} = 1.5919$ and $k_c = 6$.

```
invA = inv( A );
[ lambda1_inv_c, x1_inv, k_c ] = power_method( invA, x0, 1e-6, 100 );
lambda_n_c = 1 / lambda1_inv_c
%   lambda_n_c =
%           1.5919
k_c
%   k_c =
%           6
```

- d) We verify that we can obtain the same result of point c) by means of the following commands. We select a shift value $s = 5$ for which we can calculate $\lambda_n^{(k_c)}$ by firstly computing $\lambda_{s,1}^{(k_c)}$.

```
shift = 5;
[ lambda_s, x, k_c ] = power_method( A - shift * eye( n ), x0, 1e-6, 100 );
lambda_n_c = lambda_s + shift
%   lambda_n_c =
%           1.5919
k_c
%   k_c =
%           13
```

Solution III (MATLAB)

- a) We consider the following MATLAB commands similarly to Exercise 1, point a). The matrices M and $\tilde{M}$ are nonsingular by construction.

```
KK = 4e2; m = 1;
nnodes = 29; % number of nodes of the bridge
[ A ] = bridge_stiffness_matrix( nnodes, KK );
n = size( A, 2 );
M = m * speye( n, n );
nr = n - 3;
ir = [ 3 : n - 1 ];
Ar = A( ir, ir );
Mr = M( ir, ir );
K_Ar = condest( Ar )
%   K_Ar =
%           1.8838e+04
K_Mr = condest( Mr )
%   K_Mr =
%           1
```

- b) We compute the 10 smallest eigenvalues $\tilde{\lambda}_i$ and eigenmodes $\mathbf{x}_i$ (from $\tilde{\mathbf{x}}_i$), for $i = 1, \dots, n$, as follows.

```
neig = 10;
```

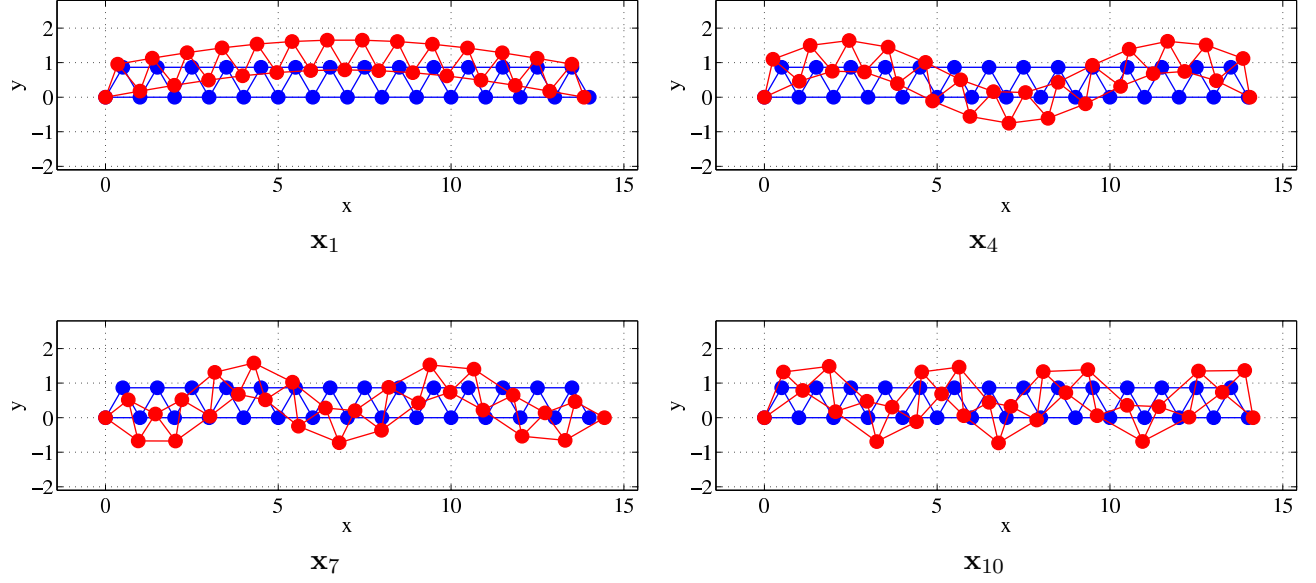


Figure 4: Eigenmodes $\mathbf{x}_1$, $\mathbf{x}_4$, $\mathbf{x}_7$, and $\mathbf{x}_{10}$.

```
[ Xr, Lr ] = eigs( Ar, Mr, neig, 'SM' );
lambda_r = diag( Lr )'
%   lambda_r =
%   0.1758    2.2183    4.3588   10.3889   24.5227   38.0159
%   49.2831   79.0215  102.4560  120.2148
X = zeros( n, neig );
X( ir, : ) = Xr( :, : );
```

We observe that L_r is a diagonal sparse matrix of size 10×10 containing the eigenvalues on the diagonal. Similarly, X_r is the matrix containing in each column the eigenvector $\tilde{\mathbf{x}}_i$; the eigenmodes $\mathbf{x}_i$ are obtained from $\tilde{\mathbf{x}}_i$ by taking into account the displacement constraints.

By considering a multiplicative factor to enhance the visualization of the eigenmodes and taking into account their ordering from the variable `lambda_r`, we obtain the results reported in Figure 4 by means of the following commands.

```
viz_fact = 3; % rescale facror for visualization of the eigenmodes
% first eigenmode, lambda = 0.1758
[ fig.h ] = plot_bridge( nnodes, viz_fact * X( :, 1 ) )
% fourth eigenmode, lambda = 10.3839
[ fig.h ] = plot_bridge( nnodes, viz_fact * X( :, 4 ) )
% seventh eigenmode, lambda = 49.2831
[ fig.h ] = plot_bridge( nnodes, viz_fact * X( :, 7 ) )
% tenth eigenmode, lambda = 120.2148
[ fig.h ] = plot_bridge( nnodes, viz_fact * X( :, 10 ) )
```